Computing Temporary Equilibria using Exact Aggregation

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Abstract

We suggest a new method of approximating temporary equilibria in heterogeneous agent models. Our approach offers a significant speedup without a notable drop in accuracy relative to established methods. We demonstrate the effectiveness of our procedure by applying it to a model with heterogeneous boundedly rational agents, and comparing its performance to that of alternative methods.

JEL Classifications: C63; C68

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1 Introduction

Heterogeneous agent models based on Huggett (1993), Aiyagari (1994), and Krusell et al. (1998) have become standard in macroeconomics. A major challenge with such models is their reliance on computational methods for finding equilibrium solutions and simulating recursive dynamic equilibria, which can be time-consuming and costly (Algan et al., 2014). When simulating heterogeneous agent economies, one must solve for the temporary equilibrium given the realization of state variables at each period of the simulation. In a temporary equilibrium, agents' decisions and prices are simultaneously pinned down by market clearing conditions – in turn, these outcomes determine aggregate dynamics. Practically, this involves finding agent-level policy rules as a function of individual states and prices, then solving for equilibrium prices by applying a sequential root solver to a set of market clearing conditions that depend on the aggregate of individual policy rules as a function of prices (Bakota, 2022). As described in Den Haan et al. (2010), the repeated aggregation of individual policy rules during the computation of temporary equilibria in a simulation may take a long time, even while using standard projection methods outlined in Judd (1992) and Judd (1996) to approximate individual policy functions.

We suggest a fast new approach to approximating temporary equilibria in heterogeneous agent models that builds on existing projection methods. The key is to store the components of the aggregation of individual policy rule approximations that are independent of prices, such that each iteration of the root solver performs a minimal number of operations to compute aggregate outcomes. We find our method to significantly outperform existing aggregation methods, especially in cases with high-dimensional individual states, while avoiding notable losses in accuracy. The remainder of the paper is organized as follows. In the following section, we build up to a formulation of our aggregation method by first describing two simpler approaches. Next, we apply all three methods to a heterogeneous agent model with bounded rationality and compare their execution times as well as accuracy in approximating temporary equilibria. The final section concludes the paper.

2 Methodology

Suppose we want to find the temporary equilibrium of a heterogeneous agent economy. We have a set of demand functions $\widetilde{x}(z,P)$, which depend on individual states z with distribution Ω and prices

P. Our objective is to find the set of market-clearing prices P^* such that

$$\widetilde{X}(P^*) \equiv \int \tilde{x}(z, P^*) d\Omega(z) = \overline{X}(P^*), \tag{1}$$

where $\widetilde{X}$ and $\overline{X}$ represent aggregate demand and supply, respectively. The problem appears simple theoretically, but is slow to solve computationally. We devise a novel approach that significantly speeds up the solution process. In this section, we present three solution methods – the first two are conventional, while the final method is our contribution.

Method #1 (Naive Global Approximation): This is the most direct of all methods. Let Ω be approximated with N points $\{\bar{z}_i\}_{i \in \mathbb{N}_N}$ with corresponding weights $\{\omega_i\}_{i \in \mathbb{N}_N}$. The approximation for the demand schedule $\widetilde{X}(P) \in \mathbb{R}^J$ for a given P may be expressed as the following weighted sum:

$$\widehat{X}^j(P) \equiv \sum_i \omega_i \tilde{x}^j(\bar{z}_i, P), \tag{2}$$

where $\tilde{x}$ is derived individually for each $(\bar{z}_i, P)$ pair, and $j = 1, \dots, J$ is an index for the set of goods in the economy. Repeating the above approximation for all J goods yields $\widetilde{X}(P)$, which can then be used to solve for the equilibrium vector of prices P^* using the market clearing condition in Eq. (1). The benefit of this approach is its precision – all of the policy functions are solved directly, therefore $\widehat{X}$ and $\widetilde{X}$ are likely to be close. On the other hand, solving the policy functions N times for a given P can be computationally taxing, especially if N is large. If an M number of steps is required for a root solver to converge to an approximation of P^*, then each $\tilde{x}^j$ needs to be recomputed a total of $M \cdot N$ number of times.

Method #2 (Interpolated Global Approximation): This method adds another layer of approximation to the first. Suppose that $\tilde{x}$ is approximated by $\hat{x}$ via projection, such that the demand for the j-th good is represented by

$$\hat{x}^j(z, P) = \sum_k \sum_l c^j_{kl} \Phi^z_k(z) \Phi^P_l(P), \tag{3}$$

where Φ^P_l are the basis functions that depend on prices and Φ^z_k are the basis functions that depend on the idiosyncratic states. The approximation for the demand schedule $\widetilde{X}(P)$ may be expressed as the following weighted sum:

$$\widehat{X}^j(P) \equiv \sum_i \omega_i \hat{x}^j(\bar{z}_i, P), \tag{4}$$

where $\bar{z}_i$ and ω_i are defined as before. Once again, Eq. (4) may be used to solve for P^* in Eq. (1). The attractiveness of this approach lies in that it requires the policy function for the j-th good to be approximated only once, after which it is inputted into Eq. (4) to compute the sum. However, given a large N, the sum in Eq. (4) may still take a substantial amount of time to compute. This is especially true if the vector of idiosyncratic states z is high-dimensional, in which case each instance of a pre-computed $\hat{x}$ may take a long time execute, causing Method #1 to outpace Method #2. Furthermore, this method likely provides a less precise approximation of $\widetilde{X}$ due to the additional layer of approximation.

Method #3 (Fast Approximation): Finally, we present our new method of approximating temporary equilibria, which is based on a simple algebraic manipulation of Method #2. Notice that substituting Eq. (3) into Eq. (4) yields

$$\widehat{X}^j(P) = \sum_i \omega_i \sum_l \sum_k c^j_{kl} \Phi^z_k(z) \Phi^P_l(P)\,, \tag{5}$$

which can be rearranged by distributing ω_i, switching the order of summation, and factoring Φ^P_l in the following manner:

$$\widehat{X}^j(P) = \sum_l \Phi^P_l(P) \sum_i \omega_i \sum_k c^j_{kl} \Phi^z_k(z)\,. \tag{6}$$

Finally, letting $C^j_l \equiv \sum_i \omega_i \sum_k c^j_{kl} \Phi^z_k(z)$ allows us to express Eq. (6) as

$$\widehat{X}^j(P) = \sum_l C^j_l \Phi^P_l(P)\,. \tag{7}$$

With this formulation of $\widehat{X}$, it is sufficient to compute C^j_l once before initializing the root solver to find P^*, since C^j_l is independent of P. This approach strictly dominates Method #2 by relying on the same projections while being significantly faster – it also likely outpaces Method #1 even when z is high-dimensional, as will be shown in Section 3.

It is worth mentioning that the execution speed of a temporary equilibrium solver becomes crucial in practice when the dynamic recursive equilibrium of a heterogeneous agent economy is being simulated over some period of time – in other words, when the temporary equilibrium solver is used repeatedly (Bakota, 2022). For such applications, the speedup offered by our method is particularly attractive. The case becomes even stronger in the context of stationary recursive equilibria, in which the idiosyncratic distribution Ω is time-invariant – C^j_l can be computed at the start of a simulation and used all throughout.

3 Application

We apply each of the three methods covered in the previous Section to the Krussell-Smith model with endogenous labor supply and bounded rationality presented in Evans et al. (2023), in which agents have heterogeneous beliefs in addition to idiosyncratic productivity. We find that our method significantly outperforms the other two methods in terms of execution time, without suffering from a notable loss in accuracy.

3.1 Model Overview

Let time be discrete. The economy is populated by a continuum of agents, such that a given agent is endowed with a unit of labor per period and derives utility from consumption c and leisure l with according to the instantaneous utility function $u(c,l)$. Each agent has a unique effective unit of labor for each unit of nominal labor supplied and receives a corresponding wage that can be separated into two parts: a common aggregate component w; and an idiosyncratic efficiency component ε that is i.i.d. across the population. $\{\varepsilon\}$ is assumed to be a Markov process with time-invariant transition function Π. In each period, an agent can trade one-period claims to capital limited by the exogenous borrowing constraint $\underline{a}$, for net return r. Goods and factor markets are assumed to be competitive.

In period t, an agent holds claims a, experiences idiosyncratic efficiency ε, and faces factor prices r_t and w_t. Additionally, the agent has a vector of beliefs, ψ, which comprise the coefficients of the forecasting model used to form expectations of next period's shadow price λ_{t+1}, such that

$$\lambda_t(a,\varepsilon,\psi) \equiv (1+r_t)u_c(c_t(a,\varepsilon,\psi),l_t(a,\varepsilon,\psi)). \tag{8}$$

All agents observe some common vector of aggregates $X_t \in \mathbb{R}^n$, and condition their individual forecasts, λ_t^e, on these aggregates. Each agent forms a forecast using the following perceived law of motion (PLM):

$$\log \hat{\lambda}_t = \langle \psi, X_{t-1} \rangle , \tag{9}$$

where $\psi \in \mathbb{R}^n$ is a vector of beliefs, $X_t \in \mathbb{R}^n$ is a vector of observable aggregates, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.[1] Given factor prices r_t and w_t, each agent uses their forecast

[1] For an in-depth discussion of this PLM and how agents update their beliefs, refer to Evans and McGough (2021) and Evans et al. (2023) – for the purposes of this paper, we will treat a distribution of ψ as given to approximate a

rule to determine their period t decisions $c_t(a,\varepsilon,\psi)$, $l_t(a,\varepsilon,\psi)$, and $a_t(a,\varepsilon,\psi)$, which satisfy the following conditions:

$$u_c(c_t(a,\varepsilon,\psi),l_t(a,\varepsilon,\psi)) \geq \beta\lambda_t^e(a_t(a,\varepsilon,\psi),\varepsilon,\psi) \quad (10)$$
$$\text{and } a_t(a,\varepsilon,\psi) \geq \underline{a}, \text{ with c.s.}$$
$$u_l(c_t(a,\varepsilon,\psi),l_t(a,\varepsilon,\psi)) = u_c(c_t(a,\varepsilon,\psi),l_t(a,\varepsilon,\psi))w_t \quad (11)$$
$$a_t(a,\varepsilon,\psi) = (1+r_t)a + w_t\cdot\varepsilon\cdot(1-l_t(a,\varepsilon,\psi)) - c_t(a,\varepsilon,\psi)\,. \quad (12)$$

The representative firm rents capital k_t at real rental rate $r_t+\delta$, hires effective labor n_t at real wage w_t, and produces output under perfect competition using CRTS technology $\theta f(k,n)$, where δ is the capital depreciation rate. We take $\{\theta_t\}$ to be a stationary process that affects total factor productivity, with dynamics given by $\theta_{t+1} = \nu_t\theta_t^{\rho}$, $|\rho| < 1$, and $\{\nu_t\}$ is iid having log-normal distribution. There are no capital installation costs. Profit maximization behavior by the firm implies factors earn their marginal products:

$$w_t = \theta_t f_n(k_t,n_t) \quad \text{and} \quad r_t+\delta = \theta_t f_k(k_t,n_t)\,. \quad (13)$$

Given agent-specific states and beliefs (a,ε,ψ), and observable aggregates X_t, the conditions (10)–(12) determine agents' decision schedules in terms of prices (r_t,w_t). The realized values of prices and other endogenous aggregates are determined by market clearing, i.e. temporary equilibrium. Mechanically, this determination requires tracking the evolving distribution of agent-specific states *and* agent-specific beliefs. Let μ_t be the contemporaneous distribution of agent-states and beliefs. Then temporary equilibrium imposes that $r_t = \theta_t f_k(k_t,n_t) - \delta$ and $w_t = \theta_t f_n(k_t,n_t)$, where k_t and n_t are determined by the market clearing conditions

$$k_t = \int a\cdot\mu_t(da,d\varepsilon,d\psi) \quad \text{and} \quad n_t = \int (1-l_t(a,\varepsilon,\psi))\mu_t(da,d\varepsilon,d\psi)\,, \quad (14)$$

and θ_t is the realized TFP shock. The n_t in the above equation depends on the policy rules $l_t(a,\varepsilon,\psi)$, which, in turn, depend implicitly on current factor prices (r_t,w_t). All must be jointly determined in the temporary equilibrium as solutions to a system of non-linear equations.

temporary equilibrium.

3.2 Performance Comparison

We compare the computational performance of the three temporary equilibrium solution methods across three dimensions: (1) the execution time of the aggregation procedure – in other words, the time it takes to find the sum of individual policy rules given all necessary inputs; (2) the execution time of the multidimensional mapping characterizing the temporary equilibrium, which needs to be computed at each step of the nonlinear solver; (3) the execution time of the nonlinear solver that approximates the set of equilibrium prices. We generate samples of these execution times and present corresponding summary statistics in Table 1. Notice that our method (Method #3) offers significant speedups across all of the performance dimensions. Furthermore, notice that Method #2 is impractically slow compared to the other methods.

	Method #1	Method #2	Method #3
Aggregation	0.051674275 (0.0426596)	N/A	0.0019208712 (1.61e-5)
Temporary Eq. Mapping	0.0933271923 (0.0703002)	N/A	8.63922e-5 (2.46e-5)
Nonlinear Solver	0.9043066747 (0.7576417)	N/A	0.0021202165 (0.0012229)
Sample Size	1000	N/A	1000

Table 1: Temporary equilibrium solution methods execution times. **Note:** Mean execution time in seconds, the with minimum execution time in parentheses.

In addition to an execution speed comparison, we compare the accuracy with which our method estimates the aggregate labor supply schedule. In Fig. 1, we plot the labor supply schedules approximated by methods #1 and #3 over a large interval surrounding the steady state wage level – we exclude method #2 due to its impractical execution speed. We find that our approach offers similar accuracy as method #1, which is the most direct method of approximation temporary equilibria.

4 Conclusion

We develop a new method of approximating temporary equilibria in heterogeneous agent models by algebraically simplifying a conventional projection method. We compare the performance of

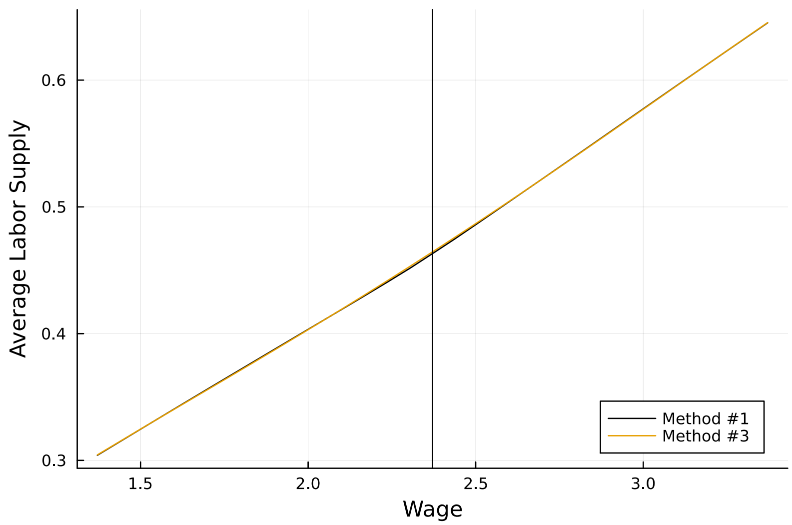

Figure 1: Aggregate labor supply as a function of wage. **Note:** The steady state wage level is represented by the vertical black line.

existing methods of approximating temporary equilibria with our method by applying them to a model with heterogeneous boundedly rational agents presented in Evans et al. (2023). We find that our method offers a significant speedup without a notable drop in accuracy.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684.

Algan, Y., Allais, O., Den Haan, W. J., and Rendahl, P. (2014). Chapter 6 - solving and simulating models with heterogeneous agents and aggregate uncertainty. In Schmedders, K. and Judd, K. L., editors, *Handbook of Computational Economics Vol. 3*, volume 3 of *Handbook of Computational Economics*, pages 277–324. Elsevier.

Bakota, I. (2022). Market clearing and krusell-smith algorithm in an economy with multiple assets. *Computational Economics*.

Den Haan, W. J., Judd, K. L., and Juillard, M. (2010). Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3. Computational Suite of Models with Heterogeneous Agents: Incomplete Markets and Aggregate Uncertainty.

Evans, D., Li, J., and McGough, B. (2023). Local rationality. *Journal of Economic Behavior & Organization*, 205:216–236.

Evans, G. W. and McGough, B. (2021). Agent-level adaptive learning.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5):953–969.

Judd, K. L. (1992). Projection methods for solving aggregate growth models. *Journal of Economic Theory*, 58(2):410–452.

Judd, K. L. (1996). Chapter 12 approximation, perturbation, and projection methods in economic analysis. In *Handbook of Computational Economics*, volume 1, pages 509–585. Elsevier.

Krusell, P., Smith, A. A., and Jr. (1998). Income and Wealth Heterogeneity in the Macroeconomy. *Journal of Political Economy*, 106(5):867–896.